BANACH SPACES WITH MANY BOUNDEDLY COMPLETE BASIC SEQUENCES FAILING PCP

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To my mother Francisca and my sister Isabel, in memoriam

ABSTRACT. We prove that there exist Banach spaces not containing ℓ_1, failing the point of continuity property and satisfying that every semi-normalized basic sequence has a boundedly complete basic subsequence. This answers in the negative the problem of the Remark 2 in [10].

1. Introduction

Recall that a Banach space is said to have the point of continuity property (PCP) provided every non-empty closed and bounded subset admits a point of continuity of the identity map from the weak to norm topologies. It is known that Banach spaces with Radon-Nikodym property, including separable dual spaces, satisfy PCP, but the converse is false (see [2]). The PCP has been characterized for separable Banach spaces in [2] and [5], and this characterization implies that Banach spaces with PCP have many boundedly complete basic sequences, and so many subspaces which are separable dual spaces. As PCP is separably determined [1], that is, a Banach space satisfies PCP if every separable subspace has PCP, it is natural looking for a sequential characterization of PCP. In this sense, it has been proved in [10] that every semi-normalized basic sequence in a Banach space with PCP has a boundedly complete subsequence. The converse of the above result is false in general, but it is open for Banach spaces not containing ℓ_1 (see Remark 2 in [10]). The goal of this note is to prove in corollary 2.4 that there exist Banach spaces failing PCP and not containing ℓ_1 such that every semi-normalized basic sequence has a boundedly complete subsequence. Concretely, the space B_∞, the natural predual of the space JT_∞, constructed in [5] is the desired example.

We begin with some notation and preliminaries. Let X be a Banach space and let $\{e_n\}$ be a basic sequence in X. $\{e_n\}$ is said to be semi-normalized

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if $0 < \inf_n \|e_n\| \leq \sup_n \|e_n\| < +\infty$, X^* denotes the topological dual of X and the closed linear span of $\{e_n\}$ is denoted by $[e_n]$. $\{e_n\}$ is called

i) *boundedly complete* provided whenever scalars $\{\lambda_i\}$ satisfy $\sup_n \|\sum_{i=1}^n \lambda_i e_i\| < +\infty$, then $\sum_n \lambda_n e_n$ converges.
ii) *shrinking* if the scalar sequence $\{\|f_{|[e_n,e_{n+1},\ldots]}\|\}$ converges to zero $\forall f \in X^*$.
iii) *supershrinking* provided $\{e_n\}$ is shrinking and whenever scalars $\{\lambda_i\}$ satisfy $\sup_n \|\sum_{i=1}^n \lambda_i e_i\| < +\infty$ and $\{\lambda_i\} \to 0$, then $\sum_n \lambda_n e_n$ converges.
iv) *strongly summing* provided is a weakly Cauchy sequence and whenever scalars $\{\lambda_i\}$ satisfy $\sup_n \|\sum_{i=1}^n \lambda_i e_i\| < +\infty$, then $\sum_n \lambda_n$ converges.

A boundedly complete basic sequence spans a dual space and a shrinking basic sequence $\{e_n\}$ spans a subspace whose dual has a basis $\{f_n\}$, called the sequence of associated functionals to $\{e_n\}$. A boundedly complete and shrinking basic sequence spans a reflexive subspace and a basic sequence in a reflexive space is both boundedly complete and shrinking [9].

The supershrinking basic sequences appear in [6] and [7], where it is proved that a Banach space X with a supershrinking basis not containing c_0 is somewhat order one quasireflexive, whenever X not contains isomorphic subspaces to c_0. Then X has many boundedly complete basic sequences. The space B_∞ has a supershrinking basis (see [6] and theorem IV.2 in [5]), not contains c_0 and fails PCP [5], so B_∞ is a good candidate to be the desired example. Other examples with a supershrinking basis are c_0 and B, the natural predual of James tree space JT [5]. It is worth to mention that a semi-normalized basis of a Banach space X is supershrinking if and only if

$$\{x^{**} \in X^{**} : \lim_n x^{**}(f_n) = 0\} = X \tag{1.1}$$

where $\{f_n\}$ is the associated functional sequence [6].

The strongly summing basic sequences appear in [11], where it is proved the remarkable c_0-theorem, which assures that every weak Cauchy non-trivial sequence in a Banach space not containing c_0, has a strongly summing basic subsequence. A weak Cauchy sequence in a Banach space is said to be non-trivial if does not converge weakly. Finally, we recall that if $\{e_n\}$ is a strongly summing sequence, then $\{v_n\}$ is a basic sequence, where $\{v_n\}$ is the diference sequence of $\{e_n\}$, that is, $v_1 = e_1$ and $v_n = e_n - e_{n-1}$ for $n > 1$ ([11]).

There is a very easy connection between supershrinking, strongly summing and boundedly complete basic sequences, which implicitly appears in [10]. We give it here for sake of completeness.

Lemma 1.1. *Let $\{e_n\}$ a semi-normalized strongly summning basic sequence with diference sequence $\{v_n\}$. If $\{v_n\}$ is supershrinking, then $\{e_n\}$ is boundedly complete. In fact, $[e_n]$ is order one quasireflexive, that is, $[e_n]$ has codimension 1 in $[e_n]^{**}$*

Proof. Let $\{\lambda_n\}$ be scalars so that $\sup_n \| \sum_{i=1}^{n} \lambda_1 e_i \| < +\infty$. We have to prove that $\sum_n \lambda_n e_n$ converges in order to obtain that $\{e_n\}$ is boundedly complete. As $\{e_n\}$ is strongly summing, hence $\sum_n \lambda_n$ converges. Define $\mu_n = \sum_{i=n}^{+\infty} \lambda_i \ \forall n$. Then $\{\mu_n\}$ converges to zero and

$$\sum_{i=1}^{n} \mu_i v_i = \sum_{i=1}^{n-1} \lambda_i e_i + \mu_n e_n \ \forall n \in \mathbb{N} \tag{1.2}$$

So, $\sup_n \| \sum_{i=1}^{n} \mu_i v_i \| < +\infty$ and then $\sum_n \mu_n v_n$ converges, by hypothesis. Finally, $\sum_n \lambda_n e_n$ converges by 1.2, since $\{\mu_n\} \to 0$.

Now, we conclude that $[e_n]$ is order one quasireflexive. For this, put $e_n^* = v_n^* - v_{n+1}^*$, where $\{v_n^*\}$ is the associated functional sequence to $\{v_n\}$. Then $\{e_n^*\}$ is the associated functional sequence to $\{e_n\}$. Observe that $[e_n]^* = [v_n^*]$, since $\{v_n\}$ is shrinking. Hence, $[e_n^*]$ has codimension 1 in $[e_n]^*$, since $x^{**}(e_n^*) = 0$ for every n and $x^{**}(v_1^*) = 1$, where $x^{**}(x^*) = \lim_n x^*(e_n)$ for every $x^* \in [e_n]^*$ exists because $\{e_n\}$ is weakly Cauchy. In fact, $[e_n]^* = [e_n^*] \oplus [v_1^*]$. But $[e_n^*]^*$ is canonically isomorphic to $[e_n]$, since $\{e_n\}$ is a boundedly complete sequence. Then $[e_n]$ has codimension 1 in $[e_n]^{**}$. ∎

2. Main results

The corollary 2.4 announced in the introduction will be deduced from the following more general result.

Theorem 2.1. *Let X be a Banach space with a semi-normalized supershrinking basis, not containing c_0. Then every non-trivial weak Cauchy sequence has a boundedly complete basic subsequence.*

Before prove this theorem, we need the following stability property of supershrinking basic block sequences.

Lemma 2.2. *Let X be a Banach space with a semi-normalized supershrinking basis $\{e_n\}$. If $v_n = \sum_{k=\sigma(n-1)+1}^{\sigma(n)} \lambda_k e_k$ is a basic block of $\{e_n\}$ with $\{\lambda_n\}$ bounded, then $\{v_n\}$ is a supershrinking basic sequence.*

Proof. Let $\{f_n\}$, $\{g_n\}$ be the sequences of associated functionals to $\{e_n\}$ and $\{v_n\}$, respectively. Then $f_k = \lambda_k g_n$ whenever $\sigma(n-1)+1 \leq k \leq \sigma(n)$. In order to show that $\{v_n\}$ is a supershrinking basic sequence we check the equality 1.1.

Pick $y^{**} \in [v_n]^{**}$ with $\lim_n y^{**}(g_n) = 0$ then $\lim_n y^{**}(f_n) = 0$, since $\{\lambda_n\}$ is bounded. So, $y^{**} \in X$ and $\{v_n\}$ is supershrinking. ∎

Now, we show that Banach spaces with a supershrinking basis without copies of c_0 contain many reflexive subspaces.

Proposition 2.3. *Let X be a Banach space with a semi-normalized supershrinking basis $\{e_n\}$ without isomorphic subspaces to c_0. Then every subsequence of $\{e_n\}$ has a further subsequence whose closed linear span is a reflexive subspace.*

Proof. It is clear that it is enough to prove that $\{e_n\}$ has a subsequence whose closed linear span is a reflexive subspace.

For this, we apply the Elton Theorem [3] to obtain $\{e_{\sigma(n)}\}$ a basic subsequence of $\{e_n\}$ such that

$$\lim_k \|\sum_{i=1}^{k} a_i e_{\sigma(i)}\| = +\infty \ \forall \{a_i\} \notin c_0.$$

We put $Y = [e_{\sigma(n)}]$. To see that Y is reflexive it suffices to prove that $\{e_{\sigma(n)}\}$ is a boundedly complete basic sequence in Y, since $\{e_{\sigma(n)}\}$ is a shrinking basic sequence.

Let $\{\lambda_n\} \subset \mathbb{R}$ such that $\sup_n \|\sum_{k=1}^{n} \lambda_k e_{\sigma(k)}\| < +\infty$. Then $\{\lambda_n\} \in c_0$ and $\sum_n \lambda_n e_{\sigma(n)}$ converges, since $\{e_{\sigma(n)}\}$ is supershrinking, that is Y is reflexive.
■

Proof. of theorem 2.1. Let $\{f_n\}$ be the functional sequence associated to $\{e_n\}$ and assume, without loss of generality that $\{e_n\}$ is monotone, that is, $\|Q_n\| \leq 1 \ \forall n \in \mathbb{N}$, where $\{Q_n = \sum_{k=1}^{n} f_k\}$ is the sequence of the projections of the basis $\{e_n\}$. Put $M = \sup_n \|e_n\|$ and let $\{x_n\}$ be a non-trivial weak Cauchy in X. By the c_0-theorem, we can assume that there is a strongly summing basic subsequence of $\{x_n\}$, so we in fact assume that $\{x_n\}$ itself is a non-trivial weak Cauchy strongly summing basic sequence.

We claim that there exist integers $0 < \sigma(1) < \sigma(2) < \ldots$, $0 = m_0 < 1 = m_1 < m_2 < \ldots$ and $\{v_n\}$ a basic sequence such that

i)

$$|f_h(x_{\sigma(n)}) - f_h(x_k)| < \frac{1}{2^{n+3} m_n M} \ \forall k \geq \sigma(n), \ h \leq m_n, \ n \in \mathbb{N} \tag{2.1}$$

ii) $v_n \in [e_k : m_{n-1} + 1 \leq k \leq m_{n+1}] \ \forall n \in \mathbb{N}$

iii) $\|v_n - z_n\| < 1/2^{n+1} \forall n \in \mathbb{N}$,

where $\{z_n\}$ is the diference sequence of $\{x_{\sigma(n)}\}$, that is, $z_1 = x_{\sigma(1)}$, $z_n = x_{\sigma(n)} - x_{\sigma(n-1)}$ for all $n > 1$.

As $\{x_n\}$ is weakly Cauchy, there is $\sigma(1) \in \mathbb{N}$ such that

$$|f_1(x_{\sigma(1)}) - f_1(x_k)| < 1/2^4 M \ \forall k \geq \sigma(1). \tag{2.2}$$

Choose $m_2 > m_1$ such that $\|\sum_{n=m_2+1}^{+\infty} f_n(x_{\sigma(1)}) e_n\| < 1/2^2$ and put $v_1 = \sum_{n=1}^{m_2} f_n(x_{\sigma(1)}) e_n$. Then $\|z_1 - v_1\| = \|\sum_{n=m_2+1}^{+\infty} f_n(x_{\sigma(1)}) e_n\| < 1/2^2$.

Pick now $\sigma(2) > \sigma(1)$ such that

$$|f_h(x_{\sigma(2)}) - f_h(x_k)| < \frac{1}{2^5 m_2 M} \ \forall k \geq \sigma(2),\ h \leq m_2 \tag{2.3}$$

Chose $m_3 > m_2$ such that $\| \sum_{n=m_3+1}^{+\infty}(f_n(x_{\sigma(2)}) - f_n(x_{\sigma(1)}))e_n\| < 1/2^4$.

Put now $v_2 = \sum_{n=m_1+1}^{m_3}(f_n(x_{\sigma(2)}) - f_n(x_{\sigma(1)}))e_n$. Then $\|z_2 - v_2\| \leq \|(f_1(x_{\sigma(2)}) - f_1(x_{\sigma(1)}))e_1\| + \| \sum_{n=m_3+1}^{+\infty}(f_n(x_{\sigma(2)}) - f_n(x_{\sigma(1)}))e_n\| < 1/2^4 + 1/2^4 = 1/2^3$, by 2.2 and 2.3.

Assume, inductively, that $m_2 < m_3 < \ldots < m_{n+1}$, $\sigma(2) < \sigma(3) < \ldots < \sigma(n)$, $v_1, v_2, \ldots, v_n$ have been constructed such that

$$|f_h(x_{\sigma(n)}) - f_h(x_k)| < \frac{1}{2^{n+3} m_n M} \ \forall k \geq \sigma(n),\ h \leq m_n \tag{2.4}$$

Pick now $m_{n+2} > m_{n+1}$ such that

$$\| \sum_{n=m_{n+2}+1}^{+\infty} (f_n(x_{\sigma(n+1)}) - f_n(x_{\sigma(n)}))e_n\| < 1/2^{n+3}. \tag{2.5}$$

Put $v_{n+1} = \sum_{i=m_n+1}^{m_{n+2}}(f_i(x_{\sigma(n+1)}) - f_i(x_{\sigma(n)}))e_i$. Then $\|z_{n+1} - v_{n+1}\| \leq \| \sum_{i=1}^{m_n}(f_i(x_{\sigma(n+1)}) - f_i(x_{\sigma(n)}))e_i\| + \| \sum_{i=m_{n+2}+1}^{+\infty}(f_i(x_{\sigma(n+1)}) - f_i(x_{\sigma(n)}))e_i\| < 1/2^{n+3} + 1/2^{n+3} = 1/2^{n+2}$, by 2.4 and 2.5.

Now, choose $\sigma(n+1) > \sigma(n)$ such that

$$|f_h(x_{\sigma(n+1)}) - f_h(x_k)| < \frac{1}{2^{n+4} m_{n+1} M} \ \forall k \geq \sigma(n+1),\ h \leq m_{n+1}. \tag{2.6}$$

Then the induction is complete and the claim is proved.

From the claim, it is clear that $\{v_n\}$ is a basic sequence equivalent to $\{z_n\}$, the diference sequence of $\{x_{\sigma(n)}\}$, since $\sum_{n=1}^{+\infty} \|z_n - v_n\| < 1/2$ (see proposition 1.a.9 in [9]). Also, we obtain that $[v_n, v_{n+1}, \ldots] \subset [e_{m_{n-1}+1}, e_{m_{n-1}+2}, \ldots]$ $\forall n \in \mathbb{N}$, so $\{v_n\}$ is a shrinking basic sequence, since $\{e_n\}$ it is.

Now, let us see that $\{v_n\}$ is a supershrinking basic sequence. For this, we chose $\{\lambda_n\}$ a scalar sequence such that $\sup_n \| \sum_{k=1}^{n} \lambda_k v_k\| < +\infty$ and we have to prove that $\sum_n \lambda_n v_n$ converges, whenever $\{\lambda_n\} \to 0$.

From the proof of the claim $v_1 = \sum_{n=1}^{m_2} f_n(x_{\sigma(1)})e_n$, and for every $n > 1$ $v_n = \sum_{k=m_{n-1}+1}^{m_{n+1}}(f_k(x_{\sigma(n)}) - f_k(x_{\sigma(n-1)}))e_k$.

Put $\mu_i = \lambda_1 f_i(x_{\sigma(1)})$ for $1 \leq i \leq m_1$, $\mu_i = \lambda_1 f_i(x_{\sigma(1)}) + \lambda_2(f_i(x_{\sigma(2)}) - f_i(x_{\sigma(1)}))$ for $m_1 + 1 \leq i \leq m_2$ and $\mu_i = \lambda_{k-1}(f_i(x_{\sigma(k-1)}) - f_i(x_{\sigma(k-2)})) + \lambda_k(f_i(x_{\sigma(k)}) - f_i(x_{\sigma(k-1)}))$ for $m_{k-1} + 1 \leq i \leq m_k$ and $k > 2$.

As $\{\lambda_n\} \to 0$, $\{e_n\}$ is a seminormalized basis of X and $\{x_n\}$ is bounded, we deduce that $\{\mu_n\} \to 0$. Furthermore, we have the following equality for all $n \in \mathbb{N}$:

$$\sum_{k=1}^{n} \lambda_k v_k = \sum_{k=1}^{m_n} \mu_k e_k + \sum_{k=m_n+1}^{m_{n+1}} \lambda_n (f_k(x_{\sigma(n)}) - f_k(x_{\sigma(n-1)}))e_k \tag{2.7}$$

Hence, whenever $m_n+1 \leq p < m_{n+1},\ n>1$ we have

$$\sum_{k=1}^{p} \mu_k e_k = \sum_{k=1}^{n} \lambda_k v_k + \sum_{k=m_n+1}^{p} \lambda_{n+1}(f_k(x_{\sigma(n+1)}) - f_k(x_{\sigma(n)}))e_k - \sum_{k=p+1}^{m_{n+1}} \lambda_n(f_k(x_{\sigma(n)}) - f_k(x_{\sigma(n-1)}))e_k \tag{2.8}$$

Now, as $\{x_n\}$ and $\{Q_n\}$ are bounded and $\{\lambda_n\} \to 0$, we obtain that

$$\lim_n \sum_{k=p+1}^{m_{n+1}} \lambda_n(f_k(x_{\sigma(n)}) - f_k(x_{\sigma(n-1)}))e_k = \lim_n \sum_{k=m_n+1}^{p} \lambda_{n+1}(f_k(x_{\sigma(n+1)}) - f_k(x_{\sigma(n)}))e_k = 0, \tag{2.9}$$

since for every $m_n+1 \leq p < m_{n+1}$, $n \in \mathbb{N},\ n>1$ we have:
(2.10)

$$\sum_{k=p+1}^{m_{n+1}} \lambda_n(f_k(x_{\sigma(n)}) - f_k(x_{\sigma(n-1)}))e_k = \lambda_n(Q_{m_{n+1}} - Q_p)(x_{\sigma(n)} - x_{\sigma(n-1)}),$$

$$\sum_{k=m_n+1}^{p} \lambda_{n+1}(f_k(x_{\sigma(n+1)}) - f_k(x_{\sigma(n)}))e_k = \lambda_{n+1}(Q_p - Q_{m_n})(x_{\sigma(n+1)} - x_{\sigma(n)})$$

From 2.8 and 2.10, it can be deduced that $\sup_p \| \sum_{n=1}^{p} \mu_n e_n \| < +\infty$ and so, $\sum_n \mu_n e_n$ converges, since $\{\mu_n\} \to 0$ and $\{e_n\}$ is supershrinking. Then $\sum_n \lambda_n v_n$ converges by 2.8 and 2.9 and we have proved that $\{v_n\}$ is a supershrinking basic sequence equivalent to the diference sequence of $\{x_{\sigma(n)}\}$. Finally, $\{x_{\sigma(n)}\}$ is boundedly complete by lemma 1.1, since it is strongly summing. In fact, $[x_{\sigma(n)}]$ is order one quasireflexive, by lemma 1.1. ∎

Corollary 2.4. *Let X be a Banach space with a semi-normalized supershrinking basis not containing c_0. Then every semi-normalized basic sequence in X has a boundedly complete subsequence spanning a reflexive or an order one quasireflexive subspace of X.*

Proof. Let $\{x_n\}$ a semi-normalized basic sequence in X. As X not contains isomorphic subspaces to ℓ_1, we can assume that $\{x_n\}$ itself is weakly Cauchy, by the ℓ_1-theorem [12]. If $\{x_n\}$ is not weakly convergent, then $\{x_n\}$ is a semi-normalized non-trivial weak Cauchy sequence and $\{x_n\}$ has a boundedly complete subsequence spanning an order one quasireflesive subspace, by theorem 2.1, and we are done.

If $\{x_n\}$ is weakly convergent, then $\{x_n\}$ converges weakly to zero, because $\{x_n\}$ is a basic sequence. Now, it is straightforward construct a subsequence of $\{x_n\}$ equivalent to a basic block of the basis. So, we can assume that $\{x_n\}$ is a semi-normalized basic sequence equivalent to a basic block of

the basis. Following the proof of proposition 1.a.11 in [9], it t is easy to construct this basic block satisfying the hypothesis of lemma 2.2. Then $\{x_n\}$ is a supershrinking basic subsequence and, by proposition 2.3, $\{x_n\}$ has a boundedly complete subsequence spanning a reflexive subspace, so we are done. ∎

As we announced in the introduction, it is enough apply the corollary 2.4 to obtain the following

Corollary 2.5. *B_∞ fails PCP, not contains isomorphic subspaces to ℓ_1 and every semi-normalized basic sequence in B_∞ has a boundedly complete subsequence spanning a reflexive or an order one quasireflexive subspace.*

Proof. The fact that B_∞ has a semi-normalized supershrinking basis is consequence of theorem IV.2 in [5]. So B_∞ has separable dual and not contains subspaces isomorphic to ℓ_1. Now, B_∞ fails PCP and not contains subspaces isomorphic to c_0 [5]. Finally, by corollary 2.4, every semi-normalized basic sequence in B_∞ has a boundedly complete subsequence spanning a reflexive or an order one quasireflexive subspace of B_∞. ∎

Let B be the natural predual of James tree space JT. It is known that B satisfies PCP, and also B has a semi-normalized supershrinking basis. (See [8] and [7]). As B not contains isomorphic subspaces to c_0, [8], we can apply the corollary 2.4, as in corollary 2.5, to obtain the following

Corollary 2.6. *Every semi-normalized basic sequence in B has a boundedly complete subsequence spanning a reflexive or an order one quasireflexive subspace of B.*

Remark 2.7. i) It has been proved in [4] that a Banach space X with separable dual satisfies PCP if, and only if, every weakly null tree in the unit sphere of X has a boundedly complete branch, which can be easily deduced from [7]. Also, it is shown in [4] that this characterization of PCP is not true for sequences, by proving that every weakly null sequence in the unit sphere of B_∞ has a boundedly complete subsequence, while B_∞ fails PCP. Hence the corollary 2.5 improves this result, since every weakly null sequence in the unit sphere of a Banach space has a semi-normalized basic subsequence.

ii) From corollary 2.4 one might think that the good sequential property in order to imply PCP for Banach spaces with separable dual is that every semi-normalized basic sequence has a subsequence spanning a reflexive subspace. And this is true, but this property implies reflexivity. Indeed, assume that X is a Banach space satisfying that every semi-normalized basic sequence has a subsequence spanning a reflexive subspace. Take a bounded sequence $\{x_n\}$ in X and prove that $\{x_n\}$ has a weakly convergent

subsequence. As X not contains subspaces isomorphic to ℓ_1, then $\{x_n\}$ has a weak Cauchy subsequence $\{y_n\}$, by the ℓ_1-theorem. If $\{y_n\}$ is not semi-normalized, then $\{y_n\}$ and so $\{x_n\}$ has a subsequence weakly convergent to zero and we are done. Hence, assume that $\{y_n\}$ is a semi-normalized weak Cauchy sequence in X. If $\{y_n\}$ is not weakly convergent, then, by the c_0-theorem, for example, $\{y_n\}$ has a semi-normalized basic subsequence, since X not contains isomorphic subspaces to c_0. By hypothesis, $\{y_n\}$ has a subsequence spanning a reflexive subspace and hence, this subsequence is weakly convergent to zero, so $\{x_n\}$ has a weakly convergent subsequence and we are done.

iii) It is known that B_∞ satisfies the convex point of continuity property CPCP [5], a weaker property than PCP. So it is natural to ask weather a Banach space satisfies CPCP, whenever every semi-normalized basic sequence has a boundedly complete subsequence.

References

[1] J. Bourgain. *Dentability and finite-dimensional decompositions.* Studia Math. 67 (1980), 135-148.

[2] J. Bourgain, H. P. Rosenthal. *Geometrical implications of certain finite dimensional decompositions*. Bull. Belg. Math. Soc. Simon Steven 32 (1980), 54-75.

[3] J. Diestel. *Sequences and series in Banach spaces.* Springer-Verlag, 1984.

[4] S. Dutta, V. P. Fonf. *On tree characterizations of G_δ-embeddings and some Banach spaces.* Israel J. Math. 167 (2008), 27-48.

[5] N. Ghoussoub, B. Maurey. *$G_\delta-$embeddings in Hilbert spaces.* J. Funct. Anal. 61 (1985), 72-97.

[6] G. López. *Banach spaces with a supershrinking basis*. Studia Math. 132 (1999), no. 1, 29–36.

[7] G. López, J. F. Mena. *RNP and KMP are equivalent for some Banach spaces with shrinking basis*. Studia Math. 118 (1996), no. 1, 11–17.

[8] J. Lindenstrauss, C. Stegall. *Examples of separable spaces which do not contain l_1 and whose duals are non-separable.* Studia Math. 54 (1975), 82-105.

[9] J. Lindenstrauss, L. Tzafriri. *Classical Banach Spaces I.* Springer Verlag. Berlin 1977.

[10] H. P. Rosenthal. *Boundedly complete weak-Cauchy basic sequences in Banach spaces with PCP*. J. Funct. Anal. 253 (2007), 772-781.

[11] H. P. Rosenthal. *A characterization of Banach spaces containing* c_0. J. Amer. Math. Soc. 7 (3) (1994), 707-748.

[12] H. P. Rosenthal. *A characterization of Banach spaces containing* ℓ_1. Proc. Natl. Acad. Sci. USA 71 (1974), 2411-2413.

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